پاسخ تمرین های تکلیف شماره۱

درس مخابرات ۲

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پاسخ تمرین ۱ :

(الف) $$P(x \geq 1) = \int_1^{\infty} \frac{1}{2} x e^{-x} dx = \frac{1}{e}$$

(ب) $$\text{Prob}(-1 < x \leq 2) = \int_{-1}^{0} -\frac{1}{2} x e^{x} dx + \int_0^2 \frac{1}{2} x e^{-x} dx = 1 - \frac{1}{e} - \frac{3}{2e^2}$$

(پ) $$\text{Prob}(x \leq -2) = \int_{-\infty}^{-2} -\frac{1}{2} x e^{x} dx = \frac{3}{2e^2}$$

(ت)

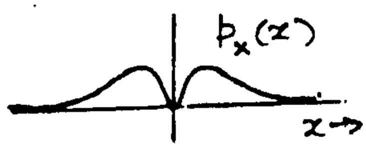
$$p_x(x) = \frac{1}{2}|x| e^{-|x|}$$

Because of even symmetry of $p_x(x)$, $\bar{x} = 0$ and

$$\bar{x}^2 = 2\int_0^{\infty} x^2 p_x(x) dx = 2\int_0^{\infty} x^2 \frac{1}{2} x e^{-x} dx$$

$$= \int_0^{\infty} x^3 e^{-x} dx = 3! = 6$$

$$\overline{x^2} = \sigma_x{}^2 + \bar{x}^2 = \sigma_x^2 = 3! = 6$$



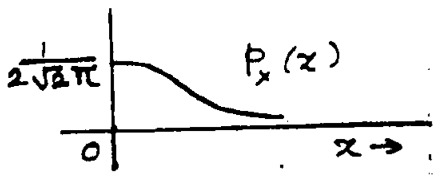
پاسخ تمرین ۲ :

(a) From the sketch it is obvious that x is not gaussian. However, it is a unilateral (rectified) version of Gaussian PDF. Hence, we can use the expression of Gaussian r.v. with a multiplier of 2.
For a gaussian r.v.

$$p_y(y) = \frac{1}{4\sqrt{2\pi}} e^{-x^2/32} \quad \text{with } \sigma_y = 4$$

(b) Hence, (I) $P(x \geq 1) = 2P(y \geq 1) = 2Q\left(\frac{1}{4}\right) = 0.8026$

(II) $P(1 < x \leq 2) = 2P(1 < y \leq 2) = 2\left[Q\left(\frac{1}{4}\right) - Q\left(\frac{2}{4}\right)\right] = 0.1856$



پاسخ تمرین ۳ :

(الف)

$f_{X,Y}(x,y)$ is a PDF so that its integral over the support region of x, y should be one.

$$\begin{aligned}
\int_0^1\int_0^1 f_{X,Y}(x,y)dxdy &= K\int_0^1\int_0^1 (x+y)dxdy \\
&= K\left[\int_0^1\int_0^1 xdxdy + \int_0^1\int_0^1 ydxdy\right] \\
&= K\left[\frac{1}{2}x^2\Big|_0^1 y\Big|_0^1 + \frac{1}{2}y^2\Big|_0^1 x\Big|_0^1\right] \\
&= K
\end{aligned}$$

Thus $K = 1$.

(ب)

$$\begin{aligned}
p(X+Y>1) &= 1 - P(X+Y\leq 1) \\
&= 1 - \int_0^1\int_0^{1-x}(x+y)dxdy \\
&= 1 - \int_0^1 x\int_0^{1-x} dydx - \int_0^1 dx\int_0^{1-x} ydy \\
&= 1 - \int_0^1 x(1-x)dx - \int_0^1 \frac{1}{2}(1-x)^2dx \\
&= \frac{2}{3}
\end{aligned}$$

(پ)

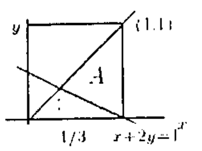
By exploiting the symmetry of $f_{X,Y}$ and the fact that it has to integrate to 1, one immediately sees that the answer to this question is 1/2. The "mechanical" solution is:

$$\begin{aligned}
p(X>Y) &= \int_0^1\int_y^1 (x+y)dxdy \\
&= \int_0^1\int_y^1 xdxdy + \int_0^1\int_y^1 ydxdy \\
&= \int_0^1 \frac{1}{2}x^2\Big|_y^1 dy - \int_0^1 yx\Big|_y^1 dy \\
&= \int_0^1 \frac{1}{2}(1-y^2)dy + \int_0^1 y(1-y)dy \\
&= \frac{1}{2}
\end{aligned}$$

(ت)

$$p(X>Y|X+2Y>1) = p(X>Y, X+2Y>1)/p(X+2Y>1)$$

The region over which we integrate in order to find $p(X>Y, X+2Y>1)$ is marked with an A in the following figure.



Thus

$$
\begin{aligned}
p(X > Y, X + 2Y > 1) &= \int_{\frac{1}{3}}^{1} \int_{\frac{1-x}{2}}^{x} (x+y)dxdy \\
&= \int_{\frac{1}{3}}^{1} \left[x(x - \frac{1-x}{2}) + \frac{1}{2}(x^2 - (\frac{1-x}{2})^2) \right] dx \\
&= \int_{\frac{1}{3}}^{1} \left(\frac{15}{8}x^2 - \frac{1}{4}x - \frac{1}{8} \right) dx \\
&= \frac{49}{108} \\
p(X + 2Y > 1) &= \int_{0}^{1} \int_{\frac{1-x}{2}}^{1} (x+y)dxdy \\
&= \int_{0}^{1} \left[x(1 - \frac{1-x}{2}) + \frac{1}{2}(1 - (\frac{1-x}{2})^2) \right] dx \\
&= \int_{0}^{1} \left(\frac{3}{8}x^2 + \frac{3}{4}x + \frac{3}{8} \right) dx \\
&= \frac{3}{8} \times \frac{1}{3}x^3\Big|_0^1 + \frac{3}{4} \times \frac{1}{2}x^2\Big|_0^1 + \frac{3}{8}x\Big|_0^1 \\
&= \frac{7}{8}
\end{aligned}
$$

Hence, $p(X > Y | X + 2Y > 1) = (49/108)/(7/8) = 14/27$

(ث)

$$F_X(x|X + 2Y > 1) = p(X \leq x, X + 2Y > 1)/p(X + 2Y > 1)$$

$$
\begin{aligned}
p(X \leq x, X + 2Y > 1) &= \int_{0}^{x} \int_{\frac{1-v}{2}}^{1} (v+y)dvdy \\
&= \int_{0}^{x} \left[\frac{3}{8}v^2 + \frac{3}{4}v + \frac{3}{8} \right] dv \\
&= \frac{1}{8}x^3 + \frac{3}{8}x^2 + \frac{3}{8}x
\end{aligned}
$$

Hence,

$$f_X(x|X + 2Y > 1) = \frac{\frac{3}{8}x^2 + \frac{6}{8}x + \frac{3}{8}}{p(X + 2Y > 1)} = \frac{3}{7}x^2 + \frac{6}{7}x + \frac{3}{7}$$

(ج)

$$
\begin{aligned}
f_X(x) &= \int_{0}^{1} (x+y)dy = x + \int_{0}^{1} ydy = x + \frac{1}{2} \\
f_Y(y) &= \int_{0}^{1} (x+y)dx = y + \int_{0}^{1} xdx = y + \frac{1}{2}
\end{aligned}
$$

پاسخ تمرین ۴:

(الف) Monotonic transformation with $g^{-1}(z) = z^2 - 1$, $dg^{-1}/dz = 2z$, $p_X(x) = ¼$ for $-1 \le x \le 3$, so

$$p_Z(z) = \frac{1}{4}\left|2z\right|[u(z) - u(z-2)] = \frac{z}{2}[u(z) - u(z-2)]$$

(ب)

$g_1(x) = -x[u(x+1) - u(x)]$, $g_1^{-1}(z) = -z[u(z) - u(z-1)]$, $dg_1^{-1}/dz = -1$

$g_2(x) = x[u(x) - u(x-3)]$, $g_2^{-1}(z) = z[u(z) - u(z-3)]$, $dg_2^{-1}/dz = 1$, $p_X(x) = ¼$ for $-1 \le x \le 3$, so

$$p_Z(z) = \begin{cases} \frac{1}{4}\left|-1\right| - \frac{1}{4}\left|1\right| = \frac{1}{2} & 0 \le z \le 1 \\ \frac{1}{4}\left|1\right| = \frac{1}{4} & 1 < z \le 3 \end{cases}$$

پاسخ تمرین ۵:

Let R_1 = "first marble is red," etc., M = "match." $P(R_1) = 5/10$, $P(W_1) = 3/10$, $P(G_1) = 2/10$,

$P(M|R_1) = P(R_2|R_1) = (5-1)/(10-1) = 4/9$, $P(M|W_1) = 2/9$, $P(M|G_1) = 1/9$

(a) $P(M) = P(M|R_1) \times P(R_1) + P(M|W_1) \times P(W_1) + P(M|G_1) \times P(G_1)$

$$= \frac{4}{9}\times\frac{5}{10}+\frac{2}{9}\times\frac{3}{10}+\frac{1}{9}\times\frac{2}{10}=\frac{14}{45}$$

(b) $P(W_1|M) = P(W_1)P(M|W_1)/P(M) = 3/14$

پاسخ تمرین ۶:

Let $Z = X + Y$. Then,

$$F_Z(z) = p(X + Y \le z) = \int_{-\infty}^{\infty}\int_{-\infty}^{z-y} f_{X,Y}(x,y)\,dx\,dy$$

Differentiating with respect to z we obtain

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty}\frac{d}{dz}\int_{-\infty}^{z-y} f_{X,Y}(x,y)\,dx\,dy \\ &= \int_{-\infty}^{\infty} f_{X,Y}(z-y,y)\frac{d}{dz}(z-y)\,dy \\ &= \int_{-\infty}^{\infty} f_{X,Y}(z-y,y)\,dy \\ &= \int_{-\infty}^{\infty} f_X(z-y)f_Y(y)\,dy \end{aligned}$$

where the last line follows from the independence of X and Y. Thus $f_Z(z)$ is the convolution of $f_X(x)$ and $f_Y(y)$. With $f_X(x) = \alpha e^{-\alpha x}u(x)$ and $f_Y(y) = \beta e^{-\beta x}u(x)$ we obtain

$$f_Z(z) = \int_0^z \alpha e^{-\alpha v}\beta e^{-\beta(z-v)}\,dv$$

If $\alpha = \beta$ then

$$f_Z(z) = \int_0^z \alpha^2 e^{-\alpha z}\,dv = \alpha^2 z e^{-\alpha z}u_{-1}(z)$$

If $\alpha \ne \beta$ then

$$f_Z(z) = \alpha\beta e^{-\beta z}\int_0^z e^{(\beta-\alpha)v}\,dv = \frac{\alpha\beta}{\beta-\alpha}\left[e^{-\alpha z} - e^{-\beta z}\right]u_{-1}(z)$$